TMA4170 Fourier Analysis

Spaces

$\mathcal{R}^2(a,b) = \{ f : \text{R.-int.} , \int_a^b |f|^2 < \infty \}$

$L^2(a,b) = \{ f : \text{measurable}, \int_a^b |f|^2 < \infty \}$

$(f,g) = \frac{1}{b-a} \int_a^b f \cdot \bar{g}$

$\| f \|^2 = \frac{1}{b-a} \int_a^b |f|^2$

Orthonormal sequences

$\{ \varphi_n \}_{n \in \mathbb{Z}}$ orthonormal if $(\varphi_n, \varphi_m) = \delta_{nm} \quad \forall n, m$

Projection of f on $\operatorname{span}\{ \varphi_n \}_{|n| \le N}$: $\quad \pi_N(f) := \sum_{|n| \le N} (f, \varphi_n) \varphi_n$

Results: $f \in \mathcal{R}^2$ (or L^2)

(a) $\| f \|^2 = \| f - \pi_N(f) \|^2 + \sum_{|n| \le N} |(f, \varphi_n)|^2$

(b) $\sum_{n \in \mathbb{Z}} |(f, \varphi_n)|^2 \le \| f \|^2$ (Bessel), $\quad (f, \varphi_n) \xrightarrow{n \to \infty} 0$ (Riemann-Lebesgue)

(c) $\| f - \pi_N(f) \| \le \| f - \sum_{|n| \le N} c_n \varphi_n \|, \quad \forall c_n \in \mathbb{C}$ (best approximation)

Fourier series

$\varphi_n = e_n = e^{inx}$ orthonormal sequence

$\hat{f}(n) = (f, e_n)$, $\quad S_N(f) = \sum_{|n| \leq N} (f, e_n)\, e_n = \pi_N(f)$

Results: $f \in \mathcal{R}^2$ (or L^2)

(a) $\|f\|^2 = \|f - S_N(f)\|^2 + \sum_{|n| \leq N} |\hat{f}(n)|^2$

(b) $\sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 \leq \|f\|^2$ Bessel inequality

(c) $\hat{f}(n) \xrightarrow{n \to \infty} 0$ Riemann-Lebesgue lemma

(d) $\|f - S_N(f)\| \leq \|f - \sum_{|n| \leq N} c_n e_n\|$, $\forall\, c_n \in \mathbb{C}$ best approximation

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Convergence of Fourier series in L^2

$f \in \mathcal{R}^2$ (or L^2) $\Rightarrow$ $\| S_N(f) - f \| \xrightarrow[N \to \infty]{} 0$

So: $f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx}$ in L^2

Plancherel: $\| f \|^2 = \sum_{n \in \mathbb{Z}} |\hat{f}|^2$ and $(f,g) = (\hat{f}, \hat{g})_{\ell^2}$

$\left[\; \| f \|^2 = \sum |\hat{f}(n)|^2 \iff \| S_N(f) - f \| \xrightarrow[N \to \infty]{} 0 \; \right]$

Local/pt. wise convergence of Fourier series

$$f(x_0 \pm) = \lim_{h \to 0^+} f(x_0 \pm h), \quad D^+ f(x_0) = \lim_{h\to 0^+} \frac{f(x_0+h) - f(x_0)}{h}, \quad D^- f(x_0) = \lim_{h \to 0^+} \frac{f(x_0) - f(x_0 - h)}{h}$$

left and right derivatives

Theorem: f Riemann-integrable (or in L^2) and $f(x_0^\pm)$, $D^\pm f(x_0)$ exist.

Then
$$S_N(f)(x_0) \longrightarrow \frac{1}{2}\left(f(x_0^-) + f(x_0^+) \right)$$

Example: